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Equivalence of unstable anharmonic oscillators and double wells

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Abstract. We give a direct and general proof of the equality of the energy levels of a pair of different quantum problems: the double well and the unstable anharmonic oscillator defined by complex translation. The 'resonances' of the two problems are also equal and are the analytic continuations of the anharmonic oscillator energy levels.

1. Introduction

Let us consider the spherically symmetric anharmonic oscillator Hamiltonian

$$H(g,d) = \tfrac{1}{2}(-\Delta + \boldsymbol{x}^2) + g^2(\boldsymbol{x}^2)^2$$

in $L^2(\boldsymbol{R}^d)$, well defined as a self-adjoint operator with discrete spectrum for any positive integer d and positive g. To any eigenvalue $E_{j,n}(0)$ of $H(0,d)$ is associated a perturbation series $\sum_{k=0} A_k(j,n)g^{2k}$, where $j = d + 2l - 2$, l is the angular momentum quantum number, and $l+1$ and n are positive integers.

We shift now to the double-well (with linear symmetry breaking) Hamiltonian

$$Q(g,j) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2(gx-1)^2 - j\left(gx - \frac{1}{2}\right)$$

in $L^2(\boldsymbol{R})$, *well defined as a self-adjoint operator with discrete spectrum for* j *real and* g positive. As above, to any eigenvalue $E_{j,n}(0)$ of $Q(0,j)$ is associated a perturbation series $\sum_{k=0} B_k(j,n)g^{2k}$ (see Simon [1] and Reed and Simon [2]).

Numerical investigations in a particular case ($j=0$) [3] and a theoretical analysis in the general case by Andrianov [4], strongly suggested the surprising identity

$$A_k(j,n) = (-1)^k B_k(j,n)$$

for $j+1$, $k+1$ and n positive integers. This identity was partially proved for $j=0$ and $k<10$ by Avron and Seiler [5].

Graffi and Grecchi in 1983 [6] and Caliceti *et al* in 1988 [7, 8] proved the Borel summability of $\sum_{k=0} B_k(j,n)(\mathrm{i}g)^{2k}$ to the eigenvalue $\tilde{E}_{j,n}(\mathrm{i}g)$ of $\tilde{Q}(\mathrm{i}g,j)$, where the tilde is used in order to distinguish the last operator from the analytic continuation of the original one from real to imaginary coupling constant. Since the series $\sum_{k=0} A_k(j,n)g^{2n}$ is also Borel summable to the eigenvalue $E_{j,n}(g)$ [9], the identity of the series implies the even more surprising identity

$$\tilde{E}_{j,n}(\mathrm{i}g) = E_{j,n}(g)$$

for $j+1$, n positive integers and g positive. Let us remark that the analytic continuation of the last identity to imaginary g gives the equality of the 'resonances' associated to the double-well and unstable anharmonic oscillator [6, 7]. One of the results of this paper is the direct proof of the last identity extended to any positive $j+2$.

The other main result of this paper is the identity of the energy levels of the double well with the ones of the (suitable definition of the) unstable anharmonic oscillator. In particular we consider the radial anharmonic oscillator Hamiltonian for imaginary coupling constant defined by complex translation

$$H_\epsilon(\mathrm{i}g, j) = \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{j^2-1}{4r_\epsilon^2} + r_\epsilon^2\right) - g^2 r_\epsilon^4$$

in $L^2(\boldsymbol{R})$, where $r_\epsilon = r - \mathrm{i}\epsilon$, for g positive and j real. Since the Hamiltonian $H_\epsilon(\mathrm{i}g, j)$ is well defined as a closed operator with discrete spectrum and its eigenvalues are independent of $\epsilon > 0$, we call them $\tilde{E}_{j,n}(\mathrm{i}g)$. Calling $E_{j,n}(g)$ the eigenvalues of $Q(g, j)$, we have the identity

$$E_{j,n}(g) = \tilde{E}_{j,n}(\mathrm{i}g)$$

for g positive, j real and n positive integer. Such identities, extended to complex parameters, are some of the few exact results in quantum anharmonic oscillators (see [1, 2]). One of their first applications is the extension to double-well 'resonances' of the distributional Borel summability recently proved for unstable anharmonic oscillator 'resonances' [10]. In order to get some physical intuition on the results, we can consider the limit operator $\tilde{H}(\mathrm{i}, 1)$ of $H_\epsilon(\mathrm{i}, 1)$ as $\epsilon \to 0$. This operator is defined by Sommerfeld and anti-Sommerfeld conditions at $\pm\infty$ respectively (both expressed by the behaviour $\psi(x) \sim x^{-1}\exp[-\mathrm{i}(x^3 - x)]$), while the corresponding 'resonance' operator $H(\mathrm{i}, 1)$ is defined only by odd parity and Sommerfeld conditions at ∞. The first operator corresponds to the motion of a particle that goes from $+\infty$ to $-\infty$ and returns to $+\infty$ when it reaches $-\infty$, while the 'resonance' operator corresponds to the motion of a particle that, after some oscilllations, escapes to ∞ in any direction. On the other hand, the potential of the unstable unharmonic oscillator on the $\boldsymbol{R}$ axis compactified to a torus is a double-well potential with a well at ∞. In this way we can justify the identity of the eigenvalues of the unstable anharmonic oscillator and double-well operator. A similar picture applies to the other identity.

The paper is organized as follows. In section 2 we give the proof of the first eigenvalue identity, and in section 3 the proof of the second one.

2. Identity of 'resonances'

The spherically symmetric anharmonic oscillator is defined by the Hamiltonian (we set g real non-negative where it is not specified)

$$H(g, d) = \tfrac{1}{2}(-\Delta + \boldsymbol{x}^2) + g^2(\boldsymbol{x}^2)^2 \tag{1}$$

in $L^2(\boldsymbol{R}^d)$, where d is an integer. An equivalent operator is obtained by a simple scaling transformation $\psi(\boldsymbol{x}) \to \lambda^{\frac{1}{2}}\psi(\lambda\boldsymbol{x})$ for $\lambda = 2^{\frac{1}{4}}$:

$$H^1(g, d) = -\Delta + \tfrac{1}{4}(\boldsymbol{x}^2 + g^2(\boldsymbol{x}^2)^2) \tag{2}$$

in $L^2(\boldsymbol{R}^d)$. Another equivalent operator is obtained by spherical coordinates and by the transform $\psi(\boldsymbol{x}) \to r^{\frac{1}{4}(1-d)}\psi(\boldsymbol{x})$, where $r = \sqrt{\boldsymbol{x}^2}$ (formally for $d = 2$):

$$H^2(g,d) = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{J_d^2 - 1}{4r^2} + \frac{r^2}{4} + \frac{g^2 r^4}{4} \tag{3}$$

in $L^2(\boldsymbol{R}^+ \times S^{d-1})$, where $J_d^2 = -4\Delta_{S^{d-1}} + (d-2)^2$, Δ_M is the Laplace–Beltrami operator on the manifold M and S^n is the n-dimensional sphere. Since the operator J_d^2 has compact resolvent with eigenvalues j^2, $j = d - 2 + 2l, l = 0, 1, 2, \ldots$, we can consider the radial operator

$$H(g,j) = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{j^2 - 1}{4r^2} + \frac{r^2}{4} + \frac{g^2 r^4}{4} \tag{4}$$

in $L^2(\boldsymbol{R}^+)$ by the boundary condition at the origin : $\psi(r) \sim r^{\frac{1}{2}(j+1)}$ for j real non-negative. Actually, for fixed g, $H(g,j)$ is an analytic family of closed (positive self-adjoint for real parameters) operators in j that can be analytically continued to all real j greater than -2. The spectrum is discrete and the eigenvalues $E_{j,n}(g)$ are real analytic for j real greater than -2. (see [2]).

The complex double-well Hamiltonian

$$\tilde{Q}(\mathrm{i}g,j) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2(\mathrm{i}gx - 1)^2 - j\left(\mathrm{i}gx - \frac{1}{2}\right) \tag{5}$$

in $L^2(\boldsymbol{R})$, for j real, is well defined as a closed operator. The operator in (5) is defined as the extension of its restriction to the Schwartz space S by the Green funtion for a parameter E external to the spectrum of the operator (see [2, 6]). The choice of the fundamental solutions $\psi_E^{+\infty}, \psi_E^{-\infty}$, is fixed by the asymptotic behaviour given by $\ln\psi_E^{\pm\infty}(x) \sim -\frac{1}{2}x^2 + \mathrm{i}\frac{1}{3}gx^3$ for $x \to \pm\infty$ respectively. These behaviours are of Gaussian type and independent of j, so that the fundamental functions are analytic in j. The Green function is also analytic in j until the parameter E is external to the spectrum, i.e. until the Wronskian of the fundamental functions is different from 0. If the spectrum of the operator (5) is not all $\boldsymbol{C}$ as in the case of the stable anharmonic oscillators, the set of values of j for which this is true is open and the family $\tilde{Q}(\mathrm{i}g,j)$ is analytic in j [2]. This is true since, as it is easy to show, the Green function is the kernel of a Hilbert–Schmidt operator [6].

Since the numerical range of the operator in (5) is all the complex plane we should prove that the spectrum is not the same set. The fundamental solutions $\psi_E^{-\infty}, \psi_E^{+\infty}$ satisfy L^2 conditions in sectors S_3, S_1 respectively, where $S_k = \{x \in \boldsymbol{C}; x \neq 0, |\arg(x) - \frac{1}{6}(2k+1)\pi| < \frac{1}{6}\pi\}$. So the operator in (5) is in the same class of equivalence $Q_{3,1}(\mathrm{i}g,j)$ [11] of the following operator:

$$Q^d(\mathrm{i}g,j) = -\frac{\mathrm{d}}{\mathrm{d}x}f(x)\frac{\mathrm{d}}{\mathrm{d}x} + \xi(x)^2(\mathrm{i}g\xi(x) - 1)^2 - j\left(\mathrm{i}g\xi(x) - \frac{1}{2}\right) - \frac{1}{4}f(x)''$$

in $L^2(\boldsymbol{R})$, obtained by distortion: $\psi(x) \to f(x)^{-\frac{1}{4}}\psi(\xi(x))$, where $\xi(x) = x\exp(\mathrm{i}\eta(x))$, $f(x)^{-\frac{1}{2}} = \xi(x)'$ and $\eta(x)$ is of class $\boldsymbol{C}^\infty$, $|x\eta(x)'|$ uniformly small, $\eta(x) \to \pm\frac{1}{6}\pi$ as $x \to \pm\infty$. It is easy to see that the numerical range of $Q^d(\mathrm{i}g,j)$ is contained in a neighbourhood of the sector $|\arg(E)| < \frac{1}{3}\pi$, so that the spectrum does not cover all the complex plane.

In analogy with the $H(\mathrm{i}g,d)$ operator we can define the operator:

$$\tilde{Q}(\mathrm{i}g,d) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2(\mathrm{i}gx - 1)^2 - J_d\left(\mathrm{i}gx - \frac{1}{2}\right) \tag{6}$$

in $\mathrm{L}^2(\boldsymbol{R} \times S^{d-1})$, $d - 1$ positive integer. We can state the following results.

Theorem 1. Let g be real non-negative and $j+2$ positive; we have the following equivalence:

$$H(g,j) \sim \tilde{Q}(\mathrm{i}g,j).$$

Corollary 2. Let g be real non-negative and $d-1$ positive integer; we have the following equivalence:

$$H(g,d) \sim \tilde{Q}(\mathrm{i}g,d).$$

Proof of theorem 1. The equivalence for $g=0$ is obvious since, in this case, both operators are exactly solvable. Now let g be positive. Since the two operator families are analytic in j, we prove the equivalence for $j \geqslant 2$ and extend it by analyticity.

As a first step we make a unitary transformation on the operator H in order to find an equivalent operator. Let $t = r^2$ and $\phi(t) = \psi(r)$; we redefine the wavefunction by $\phi \to t^{-\frac{1}{4}(j+1)}\phi(t)$ and change the space: $L^2(\boldsymbol{R}^+) \to L^2(\boldsymbol{R}^+, \frac{1}{2}t^{\frac{1}{2}j}\mathrm{d}t)$. The equivalent operator obtained is

$$K^1(g,j) = -4t\frac{\mathrm{d}^2}{\mathrm{d}t^2} - 2(2+j)\frac{\mathrm{d}}{\mathrm{d}t} + \frac{t}{4} + \frac{g^2t^2}{4} \tag{7}$$

in $L^2(\boldsymbol{R}^+, \frac{1}{2}t^{\frac{1}{2}j}\mathrm{d}t)$.

The second unitary transformation that we use is the redefinition of the wavefuntion $\phi(t) \to \phi(t)\exp(\epsilon t)$ together with the change of space $L^2(\boldsymbol{R}^+, \frac{1}{2}t^{\frac{1}{2}j}\mathrm{d}t) \to L^2(\boldsymbol{R}, \frac{1}{2}t^{\frac{1}{2}j}\exp(-2\epsilon t)\mathrm{d}t)$. The operator obtained reads

$$K^1_\epsilon(g,j) = -4t\left(\frac{\mathrm{d}}{\mathrm{d}t} - \epsilon\right)^2 - 2(2+j)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \epsilon\right) + \frac{t}{4} + \frac{g^2t^2}{4} \tag{8}$$

in $L^2(\boldsymbol{R}^+, \frac{1}{2}t^{\frac{1}{2}j}\exp(-2\epsilon t)\mathrm{d}t)$.

As a next step we simply change the space and we claim that the same formal operator as in (8) but in the new space is equivalent to the operator in (8). The new space is simply $L^2(\boldsymbol{R})$ and the equivalence is proved by the comparison of the boundary conditions in the two cases. For any value of E the formal eigenfunctions of the operator in (8) have the behaviour at zero $\phi(t) \sim c$, or $\phi(t) \sim ct^{-\frac{1}{2}j}$, where the first behaviour is consistent with the space of (8) and with $L^2(\boldsymbol{R})$, while the second one is inconsistent with both spaces. Also at $+\infty$ both spaces allow only one and the same behaviour of the two possible ones: $\ln\phi(t) \sim \pm\frac{1}{6}gt^{\frac{3}{2}}$. The behaviour at $-\infty$ is irrelevant for the first space and is also always consistent with the second one: $|\phi(-t)|^2 \sim ct^{-\frac{1}{2}(j+3)}\exp(-2\epsilon t)$. Hence we have another equivalent operator

$$K^2_\epsilon(g,j) = -4t\left(\frac{\mathrm{d}}{\mathrm{d}t} - \epsilon\right)^2 - 2(2+j)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \epsilon\right) + \frac{t}{4} + \frac{g^2t^2}{4} \tag{9}$$

in $L^2(\boldsymbol{R})$.

Now we make one of the main steps: we make the Fourier transform that is unitary and changes all the analytic, exponentially decreasing eigenfunctions of $K^2_\epsilon(g,j)$ in all the exponentially decreasing, analytic eigenfuntions of the new equivalent operator

$$Q^1_\epsilon(\mathrm{i}g,j) = -\frac{g^2}{4}\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \mathrm{i}\left(4s_\epsilon^2 + \frac{1}{4}\right)\frac{\mathrm{d}}{\mathrm{d}s} - 2(j-2)\mathrm{i}s_\epsilon \tag{10}$$

in $L^2(\boldsymbol{R})$, where $s_\epsilon = s - \mathrm{i}\epsilon$. Indeed the eigenfuntions of the last operator in (10) have the following asymptotic behaviour at infinity: $\ln\psi(s) \sim 2B(s_\epsilon)$, where $B(s) = (\mathrm{i}/6g^2)(16s^3 + 3s)$, because the other possible behaviour: $\psi(s) \sim s^{\frac{1}{2}(j-2)}$ is incompatible with the space for $j \geqslant 1$.

The last main step in the proof is the elimination of the first derivative in the operator in (10) by a change of the space and of the function: $\psi(s) \to \psi(s)\exp(B(s_\epsilon))$. Thus we obtain the equivalent operator

$$Q^2_\epsilon(\mathrm{i}g, j) = -\frac{g^2}{4}\frac{\mathrm{d}^2}{\mathrm{d}s^2} - g^2\left(\frac{4s_\epsilon^2}{g^2} + \frac{1}{4g^2}\right)^2 - 2\mathrm{i}js_\epsilon \tag{11}$$

in $L^2(\boldsymbol{R}, \exp(2B(s_\epsilon))\mathrm{d}s)$. Actually the analysis of the possible behaviours of the formal eigenfunctions shows the equivalence of the former operator to a new one, $Q^3_\epsilon(\mathrm{i}g, j)$, differing only in the space which now is $L^2(\boldsymbol{R})$. Changing the variable $s \to y = (2/g)s$ and setting $\eta = (2g)\epsilon$, we have the equivalent operator

$$Q^3_\eta(\mathrm{i}g, j) = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} - g^2\left(y_\eta^2 + \frac{1}{4g^2}\right)^2 - \mathrm{i}gjy_\eta \tag{12}$$

in $L^2(\boldsymbol{R})$, where $y_\eta = y - \mathrm{i}\eta$. As the last step it is sufficient to set $\eta = 1/2g$ in order to obtain the operator $\tilde{Q}(\mathrm{i}g, j)$. The proof of theorem 1 is finished.

Corollary 2 is an obvious consequence of theorem 1.

Remark 1. In this paper, two operators having the same spectrum are defined to be 'equivalent operators'. The equivalence of theorem 1 can be extended by analytic continuation in g and j. While $\tilde{Q}(\mathrm{i}g, j)$ is an entire family in j, the very space condition restricting the family $H(g, j)$ to $\mathrm{Re}(j) > -2$. Concerning the analyticity in g, it is well known (see [1, 2]) that by the simple scaling $x \to g^{-\frac{1}{3}}x$ we obtain entire families of operators in the new variable $\alpha = g^{-\frac{2}{3}}$. Indeed we have

$$H(g, j) \sim \alpha^{-1}H^a(\alpha, j)$$

where

$$H^a(\alpha, j) = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{j^2-1}{4r^2} + \frac{\alpha^2r^2}{4} + \frac{r^2}{4}$$

in $L^2(\boldsymbol{R}^+)$. By the same scaling on the operator in (12) we get

$$Q^3(\mathrm{i}g, j) \sim \alpha^{-1}Q^{3a}_\eta(\alpha, j)$$

where

$$Q^{3a}_\eta(\alpha, j) = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \left(y_\eta^2 + \frac{\alpha^2}{4}\right)^2 - \mathrm{i}jy_\eta \tag{13}$$

in $L^2(\boldsymbol{R})$, where $y_\eta = y - \mathrm{i}\eta$, and the last operator is defined as the one given in (5).

Taking into account remark 1 and theorem 1 we have the following equivalence.

Theorem 3.

$$H^a(\alpha, j) \sim Q_\eta^{3a}(\alpha, j) \tag{14}$$

for α, j, η complex, $\operatorname{Re} j > -2$, $\operatorname{Re} \eta > 0$. In particular we have equivalence (14) at $\alpha = 0$ corresponding to $g = \infty$, in agreement with a conjecture of Graffi and Grecchi for $j = 0$ [6].

Remark 2. For α, j, η real, $Q_\eta^{3a}(\alpha, j)$ is not self-adjoint but it is unitarily equivalent to its adjoint. More precisely, if we call T the anti-unitary operator $T\psi(y) = \overline{\psi}(y)$, and P the unitary operator $P\psi(y) = \psi(-y)$, we have

$$Q_\eta^{3a*}(\alpha, j) = TQ_\eta^{3a}(\alpha, j)T = PQ_\eta^{3a}(\alpha, j)P = Q_{-\eta}^{3a}(\alpha, -j).$$

Moreover, we have $TPQ_\eta^{3a}(\alpha, j) = Q_\eta^{3a}(\alpha, j)TP$.

Returning to the g variable on the imaginary axis, we consider the 'resonance' operators:

Corollary 4. If g and j are as in theorem 1, we have

$$H(\pm \mathrm{i}g, j) \sim \tilde{Q}(\mp g, j)$$

so that the 'resonances' are equal:

$$E_{j,n}(\pm \mathrm{i}g) = \tilde{E}_{j,n}(\mp g)$$

for $n = 1, 2, \ldots$, where $E_{j,n}(\pm \mathrm{i}g)$ are uniquely related by analytic continuation to $E_{j,n}(g)$ for g small or for g large because of stability results in the two limits (see [1, 2]).

3. Identity of energy levels

The unstable anharmonic oscillator has the formal radial Hamiltonian

$$H^1(\mathrm{i}g, j) = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{j^2 - 1}{4r^2} + \frac{r^2}{4} - \frac{g^2 r^4}{4} \tag{15}$$

for j real and r positive. A possible definition of the operator is given by complex translations. In this way we get a family of equivalent operators defined in $L^2(\boldsymbol{R})$. Actually this family of operators has a limit for vanishing translation only for $-2 < j < 2$, because of the singular behaviour at the origin. The condition at the origin is substituted by a condition at minus infinity. The operator obtained is in any case different from the analytic continuation of the stable anharmonic oscillator, as defined above, to imaginary coupling constant $H(\mathrm{i}g, j)$. As is well known, and as recalled above, such an operator is defined by dilations and defines the 'resonances'. Let $r_\eta = r - \mathrm{i}\eta$; we have by complex translation the operator

$$H_\eta(\mathrm{i}g, j) = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{j^2 - 1}{4r_\eta^2} + \frac{r_\eta^2}{4} - \frac{g^2 r_\eta^4}{4} \tag{16}$$

in $L^2(\boldsymbol{R})$, for j real and η positive. We can classify all the operators defined as (16) by a regular path (without multiple points) going from infinity to infinity in different Stokes sectors keeping the origin to the left (or to the right). The Stokes asymptotic sectors are defined by $S_k = \{r \in C; r \neq 0, |\arg(r) - (2k + 1)\pi/6| < \pi/6\}$ in which formal eigenfunctions have the asymptotic behaviour $\psi_k^\pm(r) \sim r^{-1} \exp[\pm \mathrm{i}(gr^3 - r/g)]$. A regular path going from infinity in the kth sector to infinity in the hth sector keeping the origin to the left define a class of equivalent operators called $H_{k,h}^L(\mathrm{i}g, j)$. In our case the path defined by the line $r_\eta = r - \mathrm{i}\eta$, $\eta > 0$ fixed and r varying on all $\boldsymbol{R}$, goes from the boundary of the sector S_4 to the boundary of the sector S_6 keeping the origin to the left. Our operators in (16) belong to the class $H_{4,6}^L(\mathrm{i}g, j)$ since the fundamental solutions are $\psi^{-\infty}(r) = \psi_4^-(r_\eta) \sim r^{-1} \exp[-\mathrm{i}(gr_\eta^3 - (1/g)r_\eta)]$ as $r \to -\infty$, and $\psi^\infty(r) = \psi_6^-(r_\eta) \sim r_\eta^{-1} \exp[-\mathrm{i}(gr_\eta^3 - (1/g)r_\eta)]$ as $r \to \infty$ (see Sibuya [11]).

Remark 3. The operator defined in (13) depends only on j^2, so that we have the identification $H_\eta(\mathrm{i}g, j) = H_\eta(\mathrm{i}g, -j)$. The definition of $H_\eta(\mathrm{i}g, j)$ in (16) as a closed operator is of the same kind as $\tilde{Q}(\mathrm{i}g, j)$ in (5).

Remark 4. The operator in (16) is complex and not self-adjoint but if we call T the complex conjugation operator and P the unitary inversion operator, we have $TPH_\eta = H_\eta TP$ so that the eigenfunctions of H_η, $\psi_n(r)$, are also eigenfunctions of TP. The adjoint operator $H_\eta^*(\mathrm{i}g, j)$ is equivalent to $H_\eta(\mathrm{i}g, j)$, and more precisely we have $H_\eta^*(\mathrm{i}g, j) = H_{-\eta}(\mathrm{i}g, j) = PH_\eta(\mathrm{i}g, j)P = TH_\eta(\mathrm{i}g, j)T$.

Remark 5. For $-2 < j < 2$ existing, the limit operator $\tilde{H}(\mathrm{i}g, j)$ of H_η as $\eta \to 0$ and it is defined by the limit of the boundary conditions given above and suitable matching conditions at the origin.

We should compare the operator in (16) with the self-adjoint double-well Hamiltonian

$$Q(g, j) = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + y^2(gy-1)^2 - j\left(gy - \frac{1}{2}\right) \tag{17}$$

in $L^2(\boldsymbol{R})$ and for j real.

Theorem 5. For any g, η positive and j real we have the equivalence

$$H_\eta(\mathrm{i}g, j) \sim Q(g, j) \tag{18}$$

so that the real eigenvalues are equal:

$$\tilde{E}_{j,n}(\mathrm{i}g) = E_{j,n}(g)$$

for $n = 1, 2, \ldots$.

Proof of theorem 5. If we change variable $r \to z = r^2$ in (15), we get the formal operator

$$K^3(\mathrm{i}g, j) = -z\frac{\mathrm{d}^2}{dz^2} - 2\frac{\mathrm{d}}{\mathrm{d}z} + \frac{j^2-1}{4z^2} + \frac{z}{4} - \frac{g^2z^2}{4}$$

in $L^2(\boldsymbol{R}, 1/(2\sqrt{z})\mathrm{d}z)$. We set $z = r^2$ so that $r = \sqrt{z}$ is defined on the plane cut on the positive half-axis with $r = -\mathrm{i}$ for $z = -1$. The images of the sectors S_4, S_6 in the z plane are given by D_4, D_6, where $D_4 = \{z \in \boldsymbol{C}; z \neq 0, |\arg z + 4\pi/3| < \pi/3\}$ and $D_6 = \{z \in \boldsymbol{C}; z \neq 0, |\arg z + \pi/3| < \pi/3\}$. In full analogy with the definition of the class of operators $H_{k,h}^L(\mathrm{i}g, j)$ we define the class $K_{k,h}^L(\mathrm{i}g, j)$ using the formal operator $K^3(\mathrm{i}g, j)$ and the asymptotic sectors D_k. We have the equivalence of the two classes $H_{k,h}^L(\mathrm{i}g, j)$ and $K_{k,h}^L(\mathrm{i}g, j)$ in the obvious sense of the equivalence of the operators of the two classes. The path $z = -\mathrm{i}t_\eta = -\mathrm{i}t - \eta$, for η fixed positive and t varying in $\boldsymbol{R}$, goes from the sector D_4 to the sector D_6 keeping the origin to the left. Thus, if we change variable in (16), $r \to t$ where $(r - \mathrm{i}\eta)^2 = -\mathrm{i}t - \eta$, we obtain the equivalent operator

$$K_\eta^4(\mathrm{i}g, j) = \mathrm{i}\left(-4t_\eta\frac{\mathrm{d}^2}{\mathrm{d}t^2} - 2\frac{\mathrm{d}}{\mathrm{d}t} + \frac{j^2-1}{4t_\eta} - \frac{t_\eta}{4} - \mathrm{i}\frac{g^2t_\eta^2}{4}\right) \tag{19}$$

in $L^2(\boldsymbol{R}, \mathrm{d}t/(2\sqrt{t_\eta}))$, where $t_\eta = t - \mathrm{i}\eta$. Redefining the function: $\psi(t) \to t_\eta^{\frac{1}{4}(j+1)}\psi(t)$, we have the equivalent operator

$$K_\eta^5(\mathrm{i}g, j) = \mathrm{i}\left(-4t_\eta \frac{\mathrm{d}^2}{\mathrm{d}t^2} - 2(2+j)\frac{\mathrm{d}}{\mathrm{d}t} - \frac{t_\eta}{4} - \mathrm{i}\frac{g^2 t_\eta^2}{4}\right) \tag{20}$$

in $L^2(\boldsymbol{R}, \frac{1}{2}t_\eta^{j/2}\mathrm{d}t)$. Since the formal eigenfunctions for any value of the parameter E have only exponential, increasing or decreasing, behaviour at ∞, we have an equivalent operator

$$K_\eta^6(\mathrm{i}g, j) = \mathrm{i}\left(-4t_\eta \frac{\mathrm{d}^2}{\mathrm{d}t^2} - 2(2+j)\frac{\mathrm{d}}{\mathrm{d}t} - \frac{t_\eta}{4} - \mathrm{i}\frac{g^2 t_\eta^2}{4}\right) \tag{21}$$

in $L^2(\boldsymbol{R})$. Following the method used for the first identity we make the Fourier transform, obtaining the equivalent operator

$$Q_\eta^4(g, j) = -\frac{g^2}{4}\left(\frac{\mathrm{d}}{\mathrm{d}s} - \eta\right)^2 - \left(4s^2 - \frac{1}{4}\right)\left(\frac{\mathrm{d}}{\mathrm{d}s} - \eta\right) + 2(j-2)s \tag{22}$$

in $L^2(\boldsymbol{R})$. The asymptotic behaviour of the fundamental solutions are: at $+\infty$, $\ln(\psi^{+\infty}(s)) \sim 2A(s) + \eta s$, where $A(s) = B(-\mathrm{i}s) = -(1/6g^2)(16s^3 - 3s)$, at $-\infty$, $\psi^{-\infty}(s) \sim s^{\frac{1}{2}(j-2)}\exp(\eta s)$. Now we redefine the function $\psi(s) \to \psi(s)\exp(A(s))$, obtaining the equivalent operator

$$Q_\eta^5(g, j) = -\frac{g^2}{4}\left(\frac{\mathrm{d}}{\mathrm{d}s} - \eta\right)^2 + g^2\left(\frac{4s^2}{g^2} - \frac{1}{4g^2}\right)^2 + 2js \tag{23}$$

in $L^2(\boldsymbol{R}, \exp(2A(s))\mathrm{d}s)$. The fundamental solutions of the operator in (23) do not change if we consider the same formal operator in the different space $L^2(\boldsymbol{R})$. Indeed at $+\infty$ we have the behaviour $\ln\psi^{+\infty}(s) \sim A(s) + \eta s$ and at $-\infty$ the behaviour $\ln\psi^{-\infty}(s) \sim -A(s) + \eta s$. Also making the change of variable $s \to y = -(2/g)s$, we get the equivalent operator

$$Q_\eta^6(g, j) = -\left(\frac{\mathrm{d}}{\mathrm{d}y} - \eta\right)^2 + g^2\left(y^2 - \frac{1}{4g^2}\right)^2 - gjy \tag{24}$$

in $L^2(\boldsymbol{R})$. By the real translation $y \to y + 1/2g$ and the limit $\eta \to 0$, we have the equivalence with the operator $Q(g, j)$ and the proof is finished.

On the same line as remark 1 and theorem 3 we can extend the equivalence of theorem 5 to g large (up to ∞) and complex, as follows.

Theorem 6.

$$H_\eta^a(\alpha, j) \sim Q^a(\alpha, j) \sim Q^{6a}(\alpha, j)$$

for any α, j, η complex, $\mathrm{Re}(\eta) \neq 0$, where

$$H_\eta^a(\alpha, j) = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{j^2-1}{4r_\eta^2} + \alpha^2\frac{r_\eta^2}{4} - \frac{r_\eta^4}{4} \qquad r_\eta = r - \mathrm{i}\eta$$

$$Q^a(\alpha, j) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2(x-\alpha)^2 - j\left(x - \frac{1}{2}\alpha\right)$$

and

$$Q^{6a}(\alpha, j) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(x^2 - \frac{\alpha^2}{4}\right)^2 - jx$$

are analytic families of operators with compact resolvent in $L^2(\boldsymbol{R})$.

4. Conclusions

Of course in the same way it is possible to prove the equivalence of other classes of operators defined by different pairs of Stokes sectors.

As a comment on the results we should notice that all the transformations considered have a classical meaning and that all the operators are semiclassical in the g parameter. The equivalences obtained show the existence of hidden symmetries that could help to have a 'solution' of the anharmonic oscillator problem. For the moment we can say that it is possible to transfer to the double well the distributional Borel results of the unstable anharmonic oscillators. More direct semiclassical connections of the pairs of problems are also possible. As a last comment let us remember the relevance of the anharmonic oscillators for the understanding of the interacting quantum field theories.

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